15.(B) $\int \frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$

LHS $\int \frac{\sqrt{\frac{1}{x^2}-1}}{x^3}dx$

Let $\frac{1}{x^2}-1=t \Rightarrow -\frac{2}{x^3}dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{1}{2}dt$

$\Rightarrow -\frac{1}{2}\int \sqrt{t}\,dt = -\frac{1}{3}t^{3/2}+C = -\frac{1}{3}\left(\frac{1-x^2}{x^2}\right)^{3/2}+c = -\frac{1}{3x^3}\left(\sqrt{1-x^2}\right)^3+C$

Comparing it with RHS $A(x) = -\frac{1}{3x^3}$ and $m=3$ $\therefore$ $(A(x))^m = -\frac{1}{27x^9}$

16.(D) $\int \frac{x+1}{\sqrt{2x-1}}dx$

Let $2x-1=t^2 \Rightarrow 2dx = 2t\,dt$

$\int \frac{\frac{t^2+1}{2}+1}{t}\times t\,dt \Rightarrow \frac{1}{2}\int\left(t^2+3\right)dt$

$\frac{1}{3}\left[\frac{t^3}{3}+3t\right]+C = \frac{t^3}{6}+\frac{3}{2}t+C = \frac{(2x-1)^{3/2}}{6}+\frac{3}{2}(2x-1)^{1/2}+C = (2x-1)^{1/2}\left(\frac{x+4}{3}\right)+C$

$\therefore f(x) = \frac{x+4}{3}$.

17.(B) $I = x\cos\left(\log_e x\right)+\int \sin\left(\log_e x\right)\frac{1}{x}\cdot x\cdot dx + C$

$I = x\cos\left(\log_e x\right)+\left[x\sin\log_e x - \int\cos\left(\log_e x\right)\left(\frac{1}{x}\right)\cdot x\,dx\right]$

$I = x\cos\left(\log_e x\right)+x\sin\left(\log_e x\right) - I + C$; $I = \frac{x}{2}(\cos(\log_e x)+\sin(\log_e x)+C$

18.(A) $I = \int \frac{3x^{13}+2x^{11}}{\left(2x^4+3x^2+1\right)^4}dx = \int \frac{\left(3x^{13}+2x^{11}\right)dx}{x^{16}\left(2+\frac{3}{x^2}+\frac{1}{x^4}\right)^4}dx = \int \frac{\left(3x^{-3}+2x^{-5}\right)dx}{\left(2+3x^{-2}+x^{-4}\right)^4}$

Put $2+3x^{-2}+x^{-4}=t$; $\left(-6x^{-3}-4x^{-5}\right)dx = dt$

$\left(3x^{-3}+2x^{-5}\right)dx = -\frac{dt}{2}$; $I = \frac{-1}{2}\int\frac{dt}{t^4} = \frac{1}{6}\frac{1}{t^3}+c$; $I = \frac{x^{12}}{6\left(2x^4+3x^2+1\right)^3}+c$

19.(B) $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx \Rightarrow \int \frac{2\cdot\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \Rightarrow \int \frac{2\cdot\sin\frac{5x}{2}\cos\frac{x}{2}}{\sin x}$

$\Rightarrow \int \frac{\sin 3x+\sin 2x}{\sin x} \Rightarrow \int \frac{3\sin x-4\sin^3 x+2\sin x\cos x}{\sin x} \Rightarrow \int (3-4\sin^2 x+2\cos x)dx$

$= \int (3-2(1-\cos 2x)+2\cos x)dx = \int (2\cos 2x+2\cos x+1)dx = \sin 2x+2\sin x+x+C$

20.(B) $\int \frac{dx}{x^3\left(1+x^6\right)^{2/3}} \Rightarrow \int \frac{dx}{x^7\left(1+\frac{1}{x^6}\right)^{2/3}} \Rightarrow \left(1+\frac{1}{x^6}\right) = t$

$-\frac{6}{x^7}dx = dt \Rightarrow -\frac{1}{6}\int \frac{dt}{t^{2/3}}$

$-\frac{1}{6}\left(\frac{t^{1/3}}{\frac{1}{3}}\right) = -\frac{1}{2}\left[\left(1+\frac{1}{x^6}\right)^{1/3}\right]+C = -\frac{1}{2}\frac{\left(1+x^6\right)^{1/3}}{x^6}+C = xf\left(x\right)\left(1+x^6\right)^{1/3}$

$f\left(x\right) = -\frac{1}{2x^3}$

21.(D) $I = \int \sec^{2/3}x\,\text{cosec}x^{4/3}x\,dx = \int \frac{dx}{\sin^{4/5}x\cos^{2/5}x} = \int \frac{\sec^2 x\,dx}{(\tan x)^{4/3}}$

$\Rightarrow I = \int \frac{d(\tan x)}{\tan^{4/3}} = \frac{(\tan x)^{1-\frac{4}{3}}}{1-\frac{4}{3}}+C = -3(\tan x)^{-1/3}+C$

22.(D) $\int e^{\sec x}\left(\sec x\tan x\ f\left(x\right)+\left(\sec x\tan x+\sec^2 x\right)\right)dx = e^{\sec x}f\left(x\right)+c$

Differentiating both sides w.r.t. x

$\Rightarrow \quad e^{\sec x}\left(\sec x\tan x\ f(x)+\sec x\tan x+\sec^2 x\right) = e^{\sec x}f'\left(x\right)+f\left(x\right)e^{\sec x}\cdot\sec x\tan x$

$\Rightarrow \quad f'\left(x\right) = \sec x\tan x+\sec^2 x \quad \Rightarrow \quad f\left(x\right) = \sec x+\tan x+K$

23.(D) $\int \frac{dx}{(x^2-2x+10)^2}$

Let $x-1=t,\ dx=dt$

$\int \frac{dt}{(t^2+9)^2}$

Now $\int 1\cdot\frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\frac{t}{3}+c$ .... (1)

Applying integration by parts on L.H.S of (1)

$\frac{t}{t^2+9}+2\int \frac{t^2}{(t^2+9)^2}dt = \frac{1}{3}\tan^{-1}\frac{t}{3}+c$

$\frac{t}{t^2+9}+2\int \frac{dt}{t^2+9}-18\int \frac{dt}{(t^2+9)^2} = \frac{1}{3}\tan^{-1}\frac{t}{3}+c \quad \Rightarrow \quad \int \frac{dt}{(t^2+9)^2} = \frac{1}{54}\tan^{-1}\frac{t}{3}+\frac{3t}{54(t^2+9)}+c$

Now put $t = x-1$

$$\int \frac{dx}{(x^2-2x+10)^2} = \frac{1}{54}\left(\tan^{-1}\frac{(x-1)}{3} + \frac{3(x-1)}{x^2-2x+10}\right) + c \qquad \text{Hence } A = \frac{1}{54} \text{ and } f(x) = 3(x-1)$$

24.(C) Let $I = \int (x^2)^2 e^{-x^2} x\, dx = g(x)e^{-x^2} + c$

Let $x^2 = t,\ 2x dx = dt \quad = \frac{1}{2}\int t^2 e^{-t} dt = \frac{1}{2}\left[-e^{-t}t^2 + \int 2t\, e^{-t} dt\right] \quad = \frac{1}{2}\left[-e^{-t}.t^2 - 2t\, e^{-t} + \int 2e^{-t}\, dt\right]$

$$= \frac{1}{2}[-e^{-t}.t^2 - 2te^{-t} - 2e^{-t}] \quad = e^{-t}\left[\frac{-t^2-2t-2}{2}\right] = e^{-x^2}\left[\frac{-x^4-2x^2-2}{2}\right]$$

$$g(x) = \frac{-x^4-2x^2-2}{2} \quad \Rightarrow \quad g(-1) = \frac{-1-2-2}{2} = -\frac{5}{2}.$$

25.(D) $\int \frac{2x^3-1}{x^4+x} = \int \frac{2x^3}{x^4+x} - \int \frac{1}{x^4+x}dx \ = \frac{2}{3}\int \frac{3x^2}{x^3+x} - \int \frac{1}{x.\left(x^3+1\right)} \times \frac{x^2}{x^2}.dx \ = \frac{2}{3}\log\left(x^3 H\right) - \frac{1}{3}\int \frac{dt}{t\left(t+1\right)}$

(put $x^3 = t$)

$$\frac{2}{3}\log\left(x^3+1\right) - \frac{1}{3}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$\frac{2}{3}\log\left(x^3+1\right) - \frac{1}{3}\left[\log t - \log\left(t+1\right)\right] + c = \frac{2}{3}\log\left(x^3+1\right) - \frac{1}{3}\log\left(\frac{x^3}{1+x^3}\right) + c = \frac{1}{3}\log\left|\left(x^3+1\right)^2 \frac{x^3}{1+x^3}\right| + c$$

$$\frac{1}{3}\log\left|\left(\frac{1+x^3}{x^3}\right)^3\right| + c\ ; \quad \log\left|\frac{1+x^3}{x}\right|$$

26.(A) $\int \frac{\sin(x+\alpha)}{\sin(x-\alpha)}dx$

$x - \alpha = t \Rightarrow x = t + \alpha$ **;** $\int \frac{\sin(t+2\alpha)}{\sin t} dt$ **;** $\cos(2\alpha)t + \sin 2\alpha \ \ln|\sin t| + c$